DISTRIBUTED-ORDER FRACTIONAL CAUCHY PROBLEMS ON BOUNDED DOMAINS

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Abstract. In a fractional Cauchy problem, the usual first order time derivative is replaced by a fractional derivative. The fractional derivative models time delays in a diffusion process. The order of the fractional derivative can be distributed over the unit interval, to model a mixture of delay sources. In this paper, we provide explicit strong solutions and stochastic analogues for distributed-order fractional Cauchy problems on bounded domains with Dirichlet boundary conditions. Stochastic solutions are constructed using a non-Markovian time change of a killed Markov process generated by a uniformly elliptic second order space derivative operator.

Key words and phrases. Fractional diffusion, distributed-order Cauchy problems, hitting time, Caputo fractional derivative, stochastic solution, uniformly elliptic operator, bounded domain, boundary value problem.

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1. Introduction

This paper develops explicit strong solutions for distributed-order fractional Cauchy problems on bounded domains $D \subset \mathbb{R}^d$ with Dirichlet boundary conditions. Cauchy problems $\partial u/\partial t = Lu$ model diffusion processes. The simplest case $L = \Delta = \sum_j \partial^2 u/\partial x_j^2$ governs a Brownian motion $B(t)$ with density $u(t,x)$, for which the square root scaling $u(t,x) = t^{-1/2}u(1, t^{-1/2}x)$ pertains [19]. The fractional Cauchy problem $\partial^\beta u/\partial t^\beta = Lu$ with $0 < \beta < 1$ models anomalous sub-diffusion, in which a cloud of particles spreads slower than the square root of time [25, 26, 39, 45, 47]. When $L = \Delta$, the solution $u(t,x)$ is the density of a time-changed Brownian motion $B(E_t)$, where the non-Markovian time change $E_t = \inf\{\tau > 0 : W_\tau > t\}$ is the inverse, or first passage time, of a stable subordinator W_t with index β. The scaling $W_{ct} = c^{1/\beta} W_t$ in law implies $E_{ct} = c^\beta E_t$ in law for the inverse process, so that $u(t,x) = t^{-\beta/2}u(1, t^{-\beta/2}x)$. The process $B(E_t)$ is the long-time scaling limit of a random walk [30, 31] when the random waiting times between jumps belong to the β-stable domain of attraction. Roughly speaking, a power-law distribution of waiting times leads to a fractional time derivative in the governing equation, and its power law index equals the order of the fractional time derivative. For a uniformly elliptic operator L on a bounded domain $D \subset \mathbb{R}^d$, under suitable technical conditions and assuming Dirichlet boundary conditions, the Cauchy problem governs a Markov process $X(t)$ killed at the boundary. Then the fractional Cauchy problem governs the time-changed process $X(E_t)$ [34]. Recently, Barlow and Černý [6] obtained $B(E_t)$ as the scaling limit of a random walk in a random environment, when the transition rates are drawn from a power-law distribution with index β.

Fractional derivatives are almost as old as their integer-order cousins [36, 42]. Fractional diffusion equations are important in physics, finance, hydrology, and many other areas [21, 26, 35, 43]. Nigmatullin [39] gave a physical derivation of the fractional Cauchy problem. The mathematical study of fractional Cauchy problems was initiated by Schneider and Wyss [45] and Kochubei [25, 26]. Fractional Cauchy problems were used by Zaslavsky [47] as a model for Hamiltonian chaos. Stochastic solutions of fractional Cauchy problems are the basis for particle tracking schemes [11, 15, 49].

In some applications, the waiting times between particle jumps evolve according to a more complicated process that cannot be adequately described by a single power law. Then, a waiting time model that is conditionally power law leads to a distributed-order fractional derivative in time, defined by integrating the fractional derivative of order β against the probability distribution of the power-law index [32]. The resulting distributed-order fractional Cauchy problem provides a more flexible model for anomalous sub-diffusion. The Lévy measure of a stable subordinator with index β is integrated against the power law index distribution to define a subordinator W_t. Its inverse E_t produces a stochastic solution $X(E_t)$ of the distributed-order fractional Cauchy problem on $\mathbb{R}^d$, when $X(t)$ solves the original Cauchy problem $\partial u/\partial t = Lu$. Mild solutions and stochastic solutions for distributed-order fractional

Cauchy problems on $\mathbb{R}^d$ were developed by Meerschaert and Scheffler [32], Kovács and Meerschaert [28], and Hahn, Kobayashi and Umarov [22]. Kochubei [27] obtained strong solutions for the case $L = \Delta$, and proved the uniqueness for more general uniformly elliptic operators L.

In this paper, we extend the basic approach of [30, 31] to solve distributed-order fractional Cauchy problems with Dirichlet boundary conditions on bounded domains $D \subset \mathbb{R}^d$. Our recent paper [34] treated fractional Cauchy problems on bounded domains using eigenfunction expansions and killed Markov processes. Here, we extend that theory to the distributed-order fractional Cauchy problems, using the inverse subordinators from [28, 32], along with some deep technical results from Kochubei [27]. We construct unique classical solutions, identify the stochastic process governed by the distributed order fractional Cauchy problem, and prove existence. At the end of this paper, we discuss also some open problems in the literature, including extensions to jump processes.

2. Heat kernels on bounded domains

Let D be a bounded domain in $\mathbb{R}^d$. We denote by $C^k(D), C^{k,\alpha}(D), C^k(\bar{D})$ the space of k-times differentiable functions in D, the space of k-times differential functions with k-th derivative is Hölder continuous of index α, and the space of functions that have all the derivatives up to order k extendable continuously up to the boundary ∂D of D, respectively. We refer to [34] for a detailed discussion of these spaces and concepts in this section.

A uniformly elliptic operator in divergence form is defined on C^2 functions by

$$Lu = \sum_{i,j=1}^{d} \frac{\partial \left(a_{ij}(x)(\partial u/\partial x_i)\right)}{\partial x_j} \tag{2.1}$$

with $a_{ij}(x) = a_{ji}(x)$ and, for some $\lambda > 0$,

$$\lambda \sum_{i=1}^{n} y_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \leq \lambda^{-1} \sum_{i=1}^{n} y_i^2, \quad \forall y \in \mathbb{R}^d. \tag{2.2}$$

The operator L acts on the Hilbert space $L^2(D)$. We define the initial domain $C_0^\infty(\bar{D})$ of the operator as follows. We say that f is in $C_0^\infty(\bar{D})$ if $f \in C^\infty(\bar{D})$ and $f(x) = 0$ for all $x \in \partial D$. This condition incorporates the notion of Dirichlet boundary conditions.

If X_t is a solution to

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x_0,$$

where σ is a $d \times d$ matrix and B_t is a Brownian motion, then X_t is associated with the operator L with $a = \sigma\sigma^T$ (see Chapters 1 and 5 of Bass [7]). Define the first exit time as $\tau_D(X) = \inf\{t \geq 0 : X_t \notin D\}$. Then the semigroup defined by

$$T_D(t)f(x) = E_x[f(X_t)I(\tau_D(X)) > t)] \tag{2.3}$$

has generator L, which follows by an application of the Itô formula.

Let D be a bounded domain in $\mathbb{R}^d$ and L be a uniformly elliptic operator of divergence form with Dirichlet boundary conditions on D. Then there exists a constant Λ such that for all $x \in D$,

$$\sum_{i,j=1}^{d} |a_{ij}(x)| \leq \Lambda, \tag{2.4}$$

see, for example, Davies [17, Chapter 6]

Let $T_D(t)$ be the corresponding semigroup. Then $T_D(t)$ is an ultracontractive semigroup (even an intrinsically ultracontractive semigroup), see Corollary 3.2.8, Theorem 2.1.4, Theorem 4.2.4, and Note 4.6.10 in [16]. Every ultracontractive semigroup has a kernel for the killed semigroup on a bounded domain which can be represented as a series expansion of the eigenvalues and the eigenfunctions of L (cf. [16, Theorems 2.1.4 and 2.3.6] and [20, Theorems 8.37 and 8.38]). There exist eigenvalues $0 < \mu_1 < \mu_2 \leq \mu_3 \cdots$, such that $\mu_n \to \infty$, as $n \to \infty$, with the corresponding complete orthonormal set (in L^2) of eigenfunctions ψ_n of the operator L satisfying

$$L\psi_n(x) = -\mu_n \psi_n(x), \ x \in D : \psi_n|_{\partial D} = 0. \tag{2.5}$$

In this case,

$$p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\mu_n t} \psi_n(x) \psi_n(y)$$

is the heat kernel of the killed semigroup T_D. The series converges absolutely and uniformly on $[t_0, \infty) \times D \times D$ for all $t_0 > 0$.

Denote the Laplace transform $t \to s$ of $u(t, x)$ by

$$\tilde{u}(s, x) = \int_0^\infty e^{-st} u(t, x) dt.$$

Since we are working on a bounded domain, the Fourier transform methods in [30] are not useful. Instead, we will employ Hilbert space methods. Hence, given a complete orthonormal basis $\{\psi_n(x)\}$ on $L^2(D)$, we will call

$$\bar{u}(t, n) = \int_D \psi_n(x) u(t, x) dx,$$

and

$$\begin{aligned} \hat{u}(s, n) &= \int_D \psi_n(x) \int_0^\infty e^{-st} u(t, x) dt dx \\ &= \int_D \psi_n(x) \tilde{u}(s, x) dx \\ &= \int_0^\infty e^{-st} \bar{u}(t, x) dt \text{ (when Fubini's condition holds)} \end{aligned} \tag{2.6}$$

respectively the ψ_n and the ψ_n-Laplace transforms. Since $\{\psi_n\}$ is a complete orthonormal basis for $L^2(D)$, we can invert the ψ_n-transform to obtain

$$u(t,x) = \sum_n \bar{u}(t,n)\psi_n(x)$$

for any $t > 0$, where the above series converges in the L^2 sense (e.g., see [41, Proposition 10.8.27]).

Suppose D satisfies the uniform exterior cone condition. Let $\{X_t\}$ be a Markov process in $\mathbb{R}^d$ with generator L, and f be continuous on $\bar{D}$. Then the semigroup

(2.7)

$$T_D(t)f(x) = E_x[f(X_t)I(\tau_D(X) > t)] = \int_D p_D(t,x,y)f(y)dy = \sum_{n=1}^{\infty} e^{-\mu_n t}\psi_n(x)\bar{f}(n)$$

solves the Dirichlet initial-boundary value problem in D:

$$\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= L_D u(t,x), \quad x \in D, \ t > 0, \\
u(t,x) &= 0, \quad x \in \partial D, \\
u(0,x) &= f(x), \quad x \in D.
\end{aligned}$$

See [7, Theorem 6.3, page 177] or [16, Theorem 2.1.4].

Remark 2.1. Let $L^\infty(D) = \{f : ||f||_\infty < \infty\}$, where $||f||_\infty = \mathit{ess\ sup}\ |f|$. The eigenfunctions belong to $L^\infty(D) \cap C^\alpha(D)$ for some $\alpha > 0$, by [20, Theorems 8.15 and 8.24]. If D satisfies the *uniform exterior cone condition*, then all the eigenfunctions belong to $C^\alpha(\bar{D})$ by [20, Theorem 8.29]. If $a_{ij} \in C^\alpha(\bar{D})$ and $\partial D \in C^{1,\alpha}$, then the eigenfunctions belong to $C^{1,\alpha}(\bar{D})$ by [20, Corollary 8.36]. If $a_{ij} \in C^\infty(D)$, then each eigenfunction of L is in $C^\infty(D)$ by [20, Corollary 8.11]. If $a_{ij} \in C^\infty(\bar{D})$ and $\partial D \in C^\infty$, then each eigenfunction of L is in $C^\infty(\bar{D})$ by [20, Theorem 8.13].

In the case $L = \Delta$, the Laplacian, the corresponding Markov process is a Brownian motion. We denote the eigenvalues and the eigenfunctions of Δ on D, with Dirichlet boundary conditions, by $\{\lambda_n, \phi_n\}_{n=1}^\infty$, where $\phi_n \in C^\infty(D)$.

3. Distributed order fractional derivatives

Fractional derivatives in time are useful for physical models that involve sticking or trapping [30]. They are closely connected to random walk models with long waiting times between particle jumps [31]. The fractional derivatives are essentially convolutions with a power law. Various forms of the fractional derivative can be defined, depending on the domain of the power law kernel, and the way boundary points are handled [36, 42]. The Caputo fractional derivative [10] is defined for $0 < \beta < 1$ as

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial u(r,x)}{\partial r}\frac{dr}{(t-r)^\beta}. \tag{3.1}$$

Its Laplace transform

$$\int_0^\infty e^{-st}\frac{\partial^\beta u(t,x)}{\partial t^\beta}\,ds = s^\beta \tilde{u}(s,x) - s^{\beta-1}u(0,x) \tag{3.2}$$

incorporates the initial value in the same way as the first derivative. The Caputo derivative is useful for solving differential equations that involve a fractional time derivative [21, 40], because it naturally incorporates initial values.

Fractional time derivatives emerge in anomalous diffusion models, when particles wait a long time between jumps. In the standard model, called a continuous time random walk (CTRW), a particle waits a random time $J_n > 0$ and then takes a step of random size Y_n. For the purposes of this paper, we may assume that the two sequences of i.i.d. random variables (J_n) and (Y_n) are independent. This is called an uncoupled CTRW. The particle arrives at location $X(n) = Y_1 + \cdots + Y_n$ at time $T(n) = J_1 + \cdots + J_n$. Since $N_t = \max\{n \geq 0 : T(n) \leq t\}$ is the number of jumps by time $t > 0$, the particle location at time t is $X(N_t)$. If $EY_n = 0$ and $E[Y_n^2] < \infty$ then, as the time scale $c \to \infty$, the random walk of particle jumps has a scaling limit $c^{-1/2}X([ct]) \Rightarrow B(t)$, a standard Brownian motion. If $P(J_n > t) \sim ct^{-\beta}$ for some $0 < \beta < 1$ and $c > 0$, then the scaling limit $c^{-1/\beta}T([ct]) \Rightarrow W_t$ is a strictly increasing stable Lévy process with index β, sometimes called a stable subordinator. The jump times $T(n)$ and the number of jumps N_t are inverses $\{N_t \geq n\} = \{T(n) \leq t\}$, and it follows that the scaling limits are also inverses [31, Theorem 3.2]: $c^{-\beta}N_{ct} \Rightarrow E_t$, where

$$E_t = \inf\{\tau : W_\tau > t\}, \tag{3.3}$$

so that $\{E_t \leq \tau\} = \{W_\tau \geq t\}$. A continuous mapping argument [31, Theorem 4.2] yields the CTRW scaling limit: Heuristically, since $N_{ct} \approx c^\beta E_t$, we have $c^{-\beta/2}X(N_{[ct]}) \approx (c^\beta)^{-1/2}X(c^\beta E_t) \approx B(E_t)$, a time-changed Brownian motion. The density $u(t,x)$ of the process $B(E_t)$ solves a fractional Cauchy problem

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = a\frac{\partial^2 u(t,x)}{\partial x^2}$$

for some $a > 0$, where the order of the fractional derivative equals the index of the stable subordinator. Roughly speaking, if the probability of waiting longer than time $t > 0$ between jumps falls off like $t^{-\beta}$, then the limiting particle density solves a diffusion equation that involves a fractional time derivative of the same order β. Hence, the fractional derivatives in time model sticking or trapping of particles for long (power-law distributed) periods of time.

A more flexible model for diffusion processes can be obtained by considering a sequence of CTRW. At each scale $c > 0$, we are given i.i.d. waiting times (J_n^c) and i.i.d. jumps (Y_n^c). Assume the waiting times and jumps form triangular arrays whose row sums converge in distribution. Letting $X^c(n) = Y_1^c + \cdots + Y_n^c$ and $T^c(n) = J_1^c + \cdots + J_n^c$, we require that $X^c(cu) \Rightarrow A(t)$ and $T^c(cu) \Rightarrow W_t$ as $c \to \infty$, where the limits $A(t)$ and W_t are independent Lévy processes. Letting $N_t^c = \max\{n \geq 0 : T^c(n) \leq t\}$, the

CTRW scaling limit $X^c(N_t^c) \Rightarrow A(E_t)$ [33, Theorem 2.1]. A power-law mixture model for waiting times was proposed in [32]: Take an i.i.d. sequence of mixing variables (B_i) with $0 < B_i < 1$ and assume $P\{J_i^c > u | B_i = \beta\} = c^{-1}u^{-\beta}$ for $u \geq c^{-1/\beta}$, so that the waiting times are power laws conditional on the mixing variables. The waiting time process $T^c(cu) \Rightarrow W_t$ a nondecreasing Lévy process, or subordinator, with $\mathbb{E}[e^{-sW_t}] = e^{-t\psi_W(s)}$ and Laplace exponent

$$\psi_W(s) = \int_0^\infty (e^{-sx} - 1)\phi_W(dx). \tag{3.4}$$

The Lévy measure

$$\phi_W(t, \infty) = \int_0^1 t^{-\beta}\mu(d\beta), \tag{3.5}$$

where μ is the distribution of the mixing variable [32, Theorem 3.4 and Remark 5.1]. A computation [32, Eq. (3.18)] using $\int_0^\infty (1 - e^{-st})\beta t^{-\beta-1}dt = \Gamma(1 - \beta)s^\beta$ shows that

$$\psi_W(s) = \int_0^1 s^\beta \Gamma(1 - \beta)\mu(d\beta). \tag{3.6}$$

Then $c^{-1}N_t^c \Rightarrow E_t$, the inverse subordinator [32, Theorem 3.10]. The general infinitely divisible Lévy process limit $A(t)$ forms a strongly continuous convolution semigroup with generator L (e.g., see [3]) and the corresponding CTRW scaling limit $A(E_t)$ is the stochastic solution to the distributed order-fractional Cauchy problem [32, Eq. (5.12)]defined by

$$\mathbb{D}^{(\nu)}u(t, x) = Lu(t, x), \tag{3.7}$$

where the distributed order fractional derivative

$$\mathbb{D}^{(\nu)}u(t, x) := \int_0^1 \frac{\partial^\beta u(t, x)}{\partial t^\beta}\nu(d\beta), \quad \nu(d\beta) = \Gamma(1 - \beta)\mu(d\beta). \tag{3.8}$$

To ensure that $\mathbb{D}^{(\nu)}$ is well-defined, we impose the condition

$$\int_0^1 \frac{1}{1 - \beta}\,\mu(d\beta) < \infty \tag{3.9}$$

as in [32, Eq. (3.3)]. Since $\Gamma(x) \sim 1/x$, as $x \to 0+$, this ensures that $\nu(d\beta)$ is a finite measure on $(0, 1)$.

Using triangular array limits for CTRW allows a more flexible limit model. For example, suppose Y_i are i.i.d. with mean μ and variance σ^2, and let $Y_i^c = c^{-1}\mu + c^{-1/2}(T_i - \mu)$ so that $X^c(cu) \Rightarrow A(t)$ a Brownian motion with drift. Then the density $u(t, x)$ of the CTRW scaling limit $A(E_t)$ solves (3.7) with

$$L = -v\frac{\partial}{\partial x} + a\frac{\partial^2}{\partial x^2},$$

for some $a > 0$. A triangular array of jumps with two spatial scales, one for the mean jump and another for the deviation from the mean, is necessary to get a drift term in the limit.

Since $\phi_W(0, \infty) = \infty$ in (3.4), Theorem 3.1 in [33] implies that the inverse subordinator E_t has density

$$g(t, x) = \int_0^t \phi_W(t - y, \infty) P_{W(x)}(dy).$$

This same condition ensures also that E_t is almost surely continuous, since W_t jumps in every interval, and hence is strictly increasing. Further, it follows from the definition (3.3) that E_t is monotone nondecreasing.

The following lemma shows that $h(t, \lambda) = \mathbb{E}[e^{-\lambda E_t}]$ is an eigenfunction of the distributed-order fractional derivative $\mathbb{D}^{(\nu)}$.

Lemma 3.1. *For any $\lambda > 0$, $h(t, \lambda) = \int_0^\infty e^{-\lambda x} g(t, x)\, dx = \mathbb{E}[e^{-\lambda E_t}]$ satisfies*

$$\mathbb{D}^{(\nu)} h(t, \lambda) = -\lambda h(t, \lambda); \quad h(0, \lambda) = 1. \tag{3.10}$$

Proof. First note that $h(0, \lambda) = \mathbb{E}(1) = 1$. Using (3.2) and (3.8), compute the Laplace transform of $\mathbb{D}^{(\nu)} h(t, \lambda)$ as

$$\begin{aligned}
\int_0^\infty e^{-st}\, \mathbb{D}^{(\nu)} h(t, \lambda) dt &= \int_0^\infty e^{-st} \int_0^1 \frac{\partial^\beta h(t, \lambda)}{\partial t^\beta} \nu(d\beta) dt \\
&= \int_0^1 \int_0^\infty e^{-st} \frac{\partial^\beta h(t, \lambda)}{\partial t^\beta} dt \nu(d\beta) \\
&= \int_0^1 (s^\beta \tilde{h}(s, \lambda) - s^{\beta-1}) \nu(d\beta) \\
&= \left(\tilde{h}(s, \lambda) - \frac{1}{s} \right) \psi_W(s),
\end{aligned} \tag{3.11}$$

by applying a Fubini argument which holds because $\psi_W(s) < \infty$.

The Laplace transform of $g(t, x)$ is given by [33, Eq. (3.13)]:

$$\tilde{g}(s, x) = \int_0^\infty e^{-st} g(t, x) dt = \frac{1}{s} \psi_W(s) e^{-x \psi_W(s)}. \tag{3.12}$$

Then the double Laplace transform

$$\begin{aligned}
\tilde{h}(s,\lambda) := \int_0^\infty e^{-st}h(t,\lambda)dt &= \int_0^\infty e^{-\lambda t}\Big(\int_0^\infty e^{-\lambda x}g(t,x)dx\Big)dt \\
&= \int_0^\infty e^{-\lambda x}\Big(\int_0^\infty e^{-st}g(t,x)dt\Big)dx \\
&= \frac{\psi_W(s)}{s}\int_0^\infty e^{-(\lambda+\psi_W(s))x}dx \qquad (3.13)\\
&= \frac{\psi_W(s)}{s(\lambda+\psi_W(s))}. \qquad (3.14)
\end{aligned}$$

That is, $\tilde{h}(s,\lambda)$ satisfies

$$\lambda\tilde{h}(s,\lambda) = \Big(\frac{1}{s} - \tilde{h}(s,\lambda)\Big)\psi_W(s). \tag{3.15}$$

Since E_t has continuous paths, the dominated convergence theorem implies that $t \to \mathbb{E}[e^{-\lambda E(t)}] = h(t,\lambda)$ is a continuous function. Then (3.10) follows from (3.11), (3.15) and the uniqueness of the Laplace transform. □

Lemma 3.2. *Suppose that $\mu(d\beta) = p(\beta)d\beta$, the function $\beta \mapsto \Gamma(1-\beta)p(\beta)$ is in $C^1[0,1]$, $supp(\mu) = [\beta_0,\beta_1] \subset (0,1)$ and $\mu(\beta_1) \neq 0$. Suppose also that*

$$C(\beta_0,\beta_1,p) = \int_{\beta_0}^{\beta_1} \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta > 0. \tag{3.16}$$

Then $|\partial_t h(t,\lambda)| \le \lambda k(t)$, where

$$k(t) = [C(\beta_0,\beta_1,p)\pi]^{-1}[\Gamma(1-\beta_1)t^{\beta_1-1} + \Gamma(1-\beta_0)t^{\beta_0-1}]. \tag{3.17}$$

In this case, $h(t,\lambda)$ is a classical solution to (3.10).

Proof. Using (2.19) in Kochubei [27], which follows from inverting the Laplace transform in (3.13) of $h(t,\lambda)$, we have

$$h(t,\lambda) = \frac{-\lambda}{\pi}\int_0^\infty r^{-1}e^{-tr}\Phi(r,1)dr \tag{3.18}$$

where

$$\Phi(r,1) = \frac{\int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta}{[\int_0^1 r^\beta\cos(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta + \lambda]^2 + [\int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta]^2}.$$

First we show that $|\partial_t h(t,\lambda)| < \infty$. Note that

(3.19)
$$
\begin{aligned}
|\partial_t h(\lambda,t)| &= \left| \frac{-\lambda}{\pi} \int_0^\infty r^{-1}[\partial_t e^{-tr}]\Phi(r,1)dr \right| \\
&= \frac{\lambda}{\pi} \int_0^\infty e^{-tr}\Phi(r,1)dr \\
&= \frac{\lambda}{\pi} \int_0^\infty \frac{e^{-tr}\int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta}{[\int_0^1 r^\beta \cos(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta + \lambda]^2 + [\int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta]^2} dr \\
&\le \lambda\pi^{-1} \int_0^\infty \frac{e^{-tr}dr}{\int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta} \\
&= \lambda l(t), \ \text{(say)},
\end{aligned}
$$

where $l(t)$ is a function of t only. In the case of a simple fractional derivative this $l(t)$ is given by $Ct^{\beta-1}$.

Now,

$$
(3.20) \quad \int_0^1 \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta \ge \int_{\beta_0}^{\beta_1} \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta = C(\beta_0,\beta_1,p) > 0,
$$

by assumption (3.16).

For $r > 1$, and $\beta_0 \le \beta \le \beta_1 \le 1$, we have $r^{\beta_0} \le r^\beta \le r^{\beta_1}$ and so

$$
(3.21) \quad
\begin{aligned}
\int_{\beta_0}^{\beta_1} r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta &\ge \int_{\beta_0}^{\beta_1} r^{\beta_0} \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta \\
&= r^{\beta_0} C(\beta_0,\beta_1,p).
\end{aligned}
$$

For $0 < r \le 1$, and $\beta_0 \le \beta \le \beta_1 \le 1$, we have $r^{\beta_0} \ge r^\beta \ge r^{\beta_1}$ and so

$$
(3.22) \quad
\begin{aligned}
\int_{\beta_0}^{\beta_1} r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta &\ge \int_{\beta_0}^{\beta_1} r^{\beta_1} \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta \\
&= r^{\beta_1} C(\beta_0,\beta_1,p).
\end{aligned}
$$

Using the above facts, we obtain

$$\begin{aligned}
l(t) &= \pi^{-1}\int_0^\infty \frac{e^{-tr}dr}{\int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta} \\
&= \pi^{-1}\left[\int_0^1 \frac{e^{-tr}dr}{\int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta} + \int_1^\infty \frac{e^{-tr}dr}{\int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta}\right] \\
&\le \pi^{-1}\left[\int_0^1 \frac{e^{-tr}dr}{\int_{\beta_0}^{\beta_1} r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta} + \int_1^\infty \frac{e^{-tr}dr}{\int_{\beta_0}^{\beta_1} r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta}\right] \\
&\le [C(\beta_0,\beta_1,p)\pi]^{-1}\left[\int_0^1 r^{-\beta_1}e^{-tr}dr + \int_1^\infty r^{-\beta_0}e^{-tr}dr\right] \\
&\le [C(\beta_0,\beta_1,p)\pi]^{-1}[\Gamma(1-\beta_1)t^{\beta_1-1} + \Gamma(1-\beta_0)t^{\beta_0-1}] = k(t)
\end{aligned}$$

and so $|\partial_t h(t,\lambda)| \le \lambda k(t)$. Hence, it follows from (3.8) that

$$\begin{aligned}
|\mathbb{D}^{(\nu)}h(t,\lambda)| &\le \left|\int_0^1 \frac{\partial^\beta}{\partial t^\beta}h(t,\lambda)\Gamma(1-\beta)p(\beta)d\beta\right| \\
&\le \int_0^1 \frac{1}{\Gamma(1-\beta)}\int_0^t \left|\frac{\partial h(s,\lambda)}{\partial s}\right| \frac{ds}{(t-s)^\beta}\Gamma(1-\beta)p(\beta)d\beta \\
&\le \int_0^1 \frac{1}{\Gamma(1-\beta)}\int_0^t k(s)\frac{ds}{(t-s)^\beta}\Gamma(1-\beta)p(\beta)d\beta \\
&< \infty,
\end{aligned}$$

using (3.9) and the beta density formula. Thus, the distributed-order derivative $\mathbb{D}^{(\nu)}h(t,\lambda)$ is defined in the classical sense. □

4. Distributed order fractional Cauchy problems

Fractional Cauchy problems replace the usual first-order time derivative with its fractional analogue. In this section, we prove classical (strong) solutions to distributed-order fractional Cauchy problems $\mathbb{D}^{(\nu)}u = Lu$ on bounded domains $D \subset \mathbb{R}^d$. We give also an explicit solution formula, based on the solution of the corresponding Cauchy problem. Our methods are inspired by the approach in [30], where Laplace transforms are used to handle the fractional time derivative, and spatial derivative operators (or more generally, pseudo-differential operators) are treated with Fourier transforms. In the present paper, we use an eigenfunction expansion in place of Fourier transforms, since we are operating on a bounded domain. Our first result, Theorem 4.1, is focused on a distributed-order fractional diffusion with $L = \Delta$, and we lay out all the details of the argument in the most familiar setting. Then, in Theorem 4.6, we use separation of variables to extend this approach to uniformly elliptic generators L. In the process, we explicate the stochastic solutions in terms of killed Markov processes.

Let $D_\infty = (0,\infty)\times D$ and define

$$\begin{aligned}\mathcal{H}_\Delta(D_\infty) &\equiv \{u : D_\infty \to \mathbb{R} : \Delta u \in C(D_\infty),\\ &\quad |\partial_t u(t,x)| \le k(t)g(x),\ g\in L^\infty(D),\ t>0\},\end{aligned}$$

where $k(t)$ is defined by (3.17).

We will write $u \in C^k(\bar{D})$ to mean that for each fixed $t>0$, $u(t,\cdot)\in C^k(\bar{D})$, and $u\in C_b^k(\bar{D}_\infty)$ to mean that $u\in C^k(\bar{D}_\infty)$ and is bounded. Let $\tau_D(X)=\inf\{t\ge 0 : X(t)\notin D\}$ denote the first exit time of the stochastic process $X=\{X(t)\}$.

Theorem 4.1. *Let D be a bounded domain with $\partial D\in C^{1,\alpha}$ for some $0<\alpha<1$, and $T_D(t)$ be the killed semigroup of Brownian motion $\{X(t)\}$ on D. Let E_t be the inverse* (3.3) *of the subordinator W_t, independent of $\{X(t)\}$, with Lévy measure* (3.5). *Suppose that $\mu(d\beta)=p(\beta)d\beta$, as in Lemma 3.2, and $\mathbb{D}^{(\nu)}$ is the distributed-order fractional derivative defined by* (3.8). *Then, for any $f\in D(\Delta_D)\cap C^1(\bar{D})\cap C^2(D)$ for which the eigenfunction expansion (of Δf) with respect to the complete orthonormal basis $\{\phi_n :\ n\in\mathbb{N}\}$ converges uniformly and absolutely, the unique (classical) solution of the distributed order fractional Cauchy problem*

$$\begin{aligned}\mathbb{D}^{(\nu)}u(t,x) &= \Delta u(t,x);\quad x\in D,\ t>0\\ u(t,x) &= 0,\ x\in\partial D,\ t>0,\\ u(0,x) &= f(x),\ x\in D,\end{aligned}\tag{4.1}$$

for $u\in\mathcal{H}_\Delta(D_\infty)\cap C_b(\bar{D}_\infty)\cap C^1(\bar{D})$, is given by

$$u(t,x) = E_x[f(X(E_t))I(\tau_D(X)>E_t)]$$

$$= \int_0^\infty T_D(l)f(x)g(t,l)dl \tag{4.2}$$

$$= \sum_{n=1}^\infty \bar{f}(n)\phi_n(x)h(t,\lambda_n), \tag{4.3}$$

where $h(t,\lambda)=\mathbb{E}(e^{-\lambda E_t})=\int_0^\infty e^{-\lambda x}g(t,x)\,dx$ is the Laplace transform of E_t.

Proof. Assume that $u(t,x)$ solves (4.1). Using Green's second identity, we obtain

$$\int_D[u\Delta\phi_n-\phi_n\Delta u]dx=\int_{\partial D}\left[u\frac{\partial\phi_n}{\partial\theta}-\phi_n\frac{\partial u}{\partial\theta}\right]ds=0,$$

since $u|_{\partial D}=0=\phi_n|_{\partial D}$ and $u,\phi_n\in C^1(\bar{D})$. Hence, the ϕ_n-transform of Δu is

$$\int_D\phi_n(x)\Delta u(t,x)dx=\int_D u(t,x)\Delta\phi_n(x)dx=-\lambda_n\int_D u(t,x)\phi_n(x)dx=-\lambda_n\bar{u}(t,n),$$

as ϕ_n is the eigenfunction of the Laplacian corresponding to eigenvalue λ_n.

Next we need to show that the ϕ_n transform commutes with $\mathbb{D}^{(\nu)}$. We need to show that we can interchange derivatives and integrals as follows. Observe that

$$
\begin{aligned}
&\int_D \phi_n(x)\mathbb{D}^{(\nu)}u(t,x)dx\\
&= \int_D \phi_n(x)\int_0^1 \frac{\partial^\beta}{\partial t^\beta}u(t,x)\Gamma(1-\beta)p(\beta)d\beta dx\\
&= \int_D \phi_n(x)\int_0^1 \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial u(s,x)}{\partial s}\frac{ds}{(t-s)^\beta}\Gamma(1-\beta)p(\beta)d\beta dx\\
&= \int_D \phi_n(x)\int_0^1 \int_0^t \frac{\partial u(s,x)}{\partial s}\frac{ds}{(t-s)^\beta}p(\beta)d\beta dx\\
&= \int_0^1 \int_0^t \left(\int_D \phi_n(x)\frac{\partial}{\partial s}u(s,x)dx\right)\frac{ds}{(t-s)^\beta}p(\beta)d\beta \quad \text{(by Fubini, see below)}\\
&= \int_0^1 \int_0^t \frac{\partial}{\partial s}\left(\int_D \phi_n(x)u(s,x)dx\right)\frac{ds}{(t-s)^\beta}p(\beta)d\beta\\
&= \int_0^1 \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\partial}{\partial s}\bar{u}(s,n)\frac{ds}{(t-s)^\beta}\Gamma(1-\beta)p(\beta)d\beta\\
&= \mathbb{D}^{(\nu)}\bar{u}(s,n).
\end{aligned}
\tag{4.4}
$$

The Fubini-Tonelli argument for the interchange of order of integration in (4.4) can be justified as follows:

$$
\begin{aligned}
&\left|\int_D \phi_n(x)\mathbb{D}^{(\nu)}u(t,x)dx\right|\\
&= \left|\int_D \phi_n(x)\int_0^1 \int_0^t \frac{\partial u(s,x)}{\partial s}\frac{ds}{(t-s)^\beta}p(\beta)d\beta dx\right|\\
&\le \int_D |\phi_n(x)|\int_0^1 \int_0^t \left|\frac{\partial u(s,x)}{\partial s}\right|\frac{ds}{(t-s)^\beta}p(\beta)d\beta dx\\
&\le \int_D |\phi_n(x)||g(x)|dx\int_0^1 \int_0^t k(s)\frac{ds}{(t-s)^\beta}p(\beta)d\beta\\
&\le \sqrt{|D|}||\phi_n||_{L^2(D)}||g||_{L^\infty}\int_0^1 \int_0^t [C(\beta_0,\beta_1,p)\pi]^{-1}[\Gamma(1-\beta_1)s^{\beta_1-1}+\Gamma(1-\beta_0)s^{\beta_0-1}]\\
&\qquad\qquad \times \frac{ds}{(t-s)^\beta}p(\beta)d\beta,
\end{aligned}
$$

using (3.17). Further, using the property of beta density, for $0<\gamma,\eta<1$,

$$
\int_0^t \frac{1}{(t-s)^\gamma}s^{\eta-1}ds = t^{\eta-\gamma}\int_0^1 (1-u)^{(1-\gamma)-1}u^{\eta-1}du = B(1-\gamma,\eta)t^{\eta-\gamma},
$$

where $B(a,b)$ denotes the usual beta function. Thus,

$$
\begin{aligned}
&\left|\int_D \phi_n(x)\mathbb{D}^{(\nu)}u(t,x)dx\right| \\
&\le \sqrt{|D|}||\phi_n||_{L^2(D)}||g||_{L^\infty}[C(\beta_0,\beta_1,p)\pi]^{-1}\Bigg[\int_0^1\int_0^t \Gamma(1-\beta_1)s^{\beta_1-1}\frac{ds}{(t-s)^\beta}p(\beta)d\beta \\
&+\int_0^1\int_0^t \Gamma(1-\beta_0)s^{\beta_0-1}\frac{ds}{(t-s)^\beta}p(\beta)d\beta\Bigg] \\
&= \sqrt{|D|}||\phi_n||_{L^2(D)}||g||_{L^\infty}[C(\beta_0,\beta_1,p)\pi]^{-1}\Bigg[\Gamma(1-\beta_1)\int_0^1 t^{\beta_1-\beta}B(1-\beta,\beta_1)p(\beta)d\beta \\
&+\Gamma(1-\beta_0)\int_0^1 t^{\beta_0-\beta}B(1-\beta,\beta_0)p(\beta)d\beta\Bigg] \\
&<\infty,
\end{aligned}
$$

which justifies the use of Fubini-Tonelli theorem in (4.4).

Thus, applying the ϕ_n-transforms to (4.1), we get

$$\mathbb{D}^{(\nu)}\bar{u}(t,n) = -\lambda_n \bar{u}(t,n). \tag{4.5}$$

Since u is uniformly continuous on $C([0,\epsilon])\times\bar{D})$, it is uniformly bounded on $[0,\epsilon]\times\bar{D}$. Thus, by the dominated convergence theorem, we have $\lim_{t\to 0}\int_D u(t,x)\phi_n(x)dx = \bar{f}(n)$. Hence, $\bar{u}(0,n)=\bar{f}(n)$. A similar argument shows that $t\mapsto \bar{u}(t,n)$ is a continuous function of $t\in[0,\infty)$ for every n. Then, taking Laplace transforms on both sides of (4.5), we get

$$\int_0^1 (s^\beta \hat{u}(s,n) - s^{\beta-1}\bar{u}(0,n))\Gamma(1-\beta)p(\beta)d\beta = -\lambda_n\hat{u}(s,n) \tag{4.6}$$

which leads to

$$\hat{u}(s,n) = \frac{\bar{f}(n)\int_0^1 s^{\beta-1}\Gamma(1-\beta)p(\beta)d\beta}{\int_0^1 s^\beta\Gamma(1-\beta)p(\beta)d\beta+\lambda_n}. \tag{4.7}$$

Recall $\int_0^1 s^\beta\Gamma(1-\beta)p(\beta)d\beta = \psi_W(s)$. Then, from (4.7),

$$
\begin{aligned}
\hat{u}(s,n) &= \frac{\bar{f}(n)\psi_W(s)}{s(\psi_W(s)+\lambda_n)} \\
&= \frac{1}{s}\bar{f}(n)\psi_W(s)\int_0^\infty e^{-(\psi_W(s)+\lambda_n)l}dl \\
&= \int_0^\infty e^{-\lambda_n l}\bar{f}(n)\frac{1}{s}\psi_W(s)e^{-l\psi_W(s)}dl, \qquad (4.8)
\end{aligned}
$$

using the property of the exponential density.

The ϕ_n-transform of the killed semigroup $T_D(l)f(x) = \sum_{m=1}^{\infty} e^{-\lambda_m l}\phi_m(x)\bar{f}(m)$ from (2.7) is found as follows. Since $\{\phi_n, n \in \mathbb{N}\}$ is a complete orthonormal basis of $L^2(D)$, we get

$$
\begin{aligned}
\overline{[T_D(l)f]}(n) &= \int_D \phi_n(x) T_D(l) f(x) dx \\
&= \int_D \phi_n(x) \int_D p_D(l,x,y) f(y) dy dx \\
&= \int_D \phi_n(x) \int_D \sum_{m=1}^{\infty} e^{-\lambda_m l} \phi_m(x)\phi_m(y) f(y) dy dx \\
&= \int_D \phi_n(x) \sum_{m=1}^{\infty} e^{-\lambda_m l} \phi_m(x) \int_D \phi_m(y) f(y) dy dx \qquad (4.9)\\
&= \int_D \phi_n(x) \sum_{m=1}^{\infty} e^{-\lambda_m l} \phi_m(x) \bar{f}(m) dx \\
&= \sum_{m=1}^{\infty} e^{-\lambda_m l} \bar{f}(m) \int_D \phi_n(x)\phi_m(x) dx \\
&= e^{-l\lambda_n} \bar{f}(n).
\end{aligned}
$$

Since $T_D(t)$ is a contraction semigroup on $L^2(D)$, $T_D(t)f \in L^2(D)$ and hence Fubini-Tonelli applies.
By (3.20) in [32], we have

$$
\frac{1}{s}\psi_W(s) e^{-\psi_W(s)l} = \int_0^{\infty} e^{-st} g(t,l) dt, \qquad (4.10)
$$

where $g(t,l)$ is the smooth density of E_t.
Using the results (4.9), (4.10) and (4.8), we get

$$
\begin{aligned}
\int_0^{\infty} e^{-st} \bar{u}(t,n) dt &= \hat{u}(s,n) = \int_0^{\infty} \overline{[T_D(l)f]}(n) \left[\int_0^{\infty} e^{-st} g(t,l) dt \right] dl \\
&= \int_0^{\infty} e^{-st} \left[\int_0^{\infty} \overline{[T_D(l)f]}(n) g(t,l) dl \right] dt
\end{aligned}
$$

using (2.6).
By the uniqueness of the Laplace transform,

$$
\begin{aligned}
\bar{u}(t,n) &= \int_0^{\infty} \overline{[T_D(l)f]}(n) g(t,l) dl \\
&= \bar{f}(n) \int_0^{\infty} e^{-\lambda_n l} g(t,l) dl \ (\text{ using}(4.9)) \\
&= \bar{f}(n) h(t,\lambda_n), \qquad (4.11)
\end{aligned}
$$

where $h(t,\lambda)=\int_0^\infty e^{-l\lambda}g(t,l)dl$ is the Laplace transform of E_t.
Note that inverting the ϕ_n-transform is equal to multiplying both sides of the above equation by ϕ_n and then summing up from $n=1$ to ∞. Also, the unique inverse $u(t,\ .)\in L^2(D)$ for each fixed $t\geq 0$. Inverting the ϕ_n-transform $\bar{u}(t,n)$ in (4.11), we get an L^2-convergent solution of (4.1) as

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty}\bar{u}(t,n)\phi_n(x) \\ &= \sum_{n=1}^{\infty}\bar{f}(n)\phi_n(x)h(\lambda_n,t) \end{aligned} \tag{4.12}$$

for each $t\geq 0$.

In order to complete the proof, it will suffice to show that the series (4.12) converges pointwise, and satisfies all the conditions in (4.1).

Step 1. We begin showing that (4.12) convergence uniformly in $t\in[0,\infty)$ in the L^2 sense. Define the sequence of functions

$$u_N(t,x)=\sum_{n=1}^{N}\bar{f}(n)\phi_n(x)h(t,\lambda_n). \tag{4.13}$$

Since $g(t,l)$ is the density of E_t, we have $0<h(t,\lambda_n)\leq 1$. Also, if $0<\lambda\leq\eta$, then $h(t,\eta)\leq h(t,\lambda)$, showing that $h(t,\lambda)$ is nonincreasing in λ.

Since $f\in L^2(D)$, we can write $f(x)=\sum_{n=1}^{\infty}\bar{f}(n)\phi_n(x)$, and then the Parseval identity yields

$$\sum_{n=1}^{\infty}(\bar{f}(n))^2=||f||_{2,D}^2<\infty.$$

Then, given $\epsilon>0$, we can choose $n_0(\epsilon)$ such that

$$\sum_{n=n_0(\epsilon)}^{\infty}(\bar{f}(n))^2<\epsilon. \tag{4.14}$$

For $N>M>n_0(\epsilon)$ and $t\geq 0$,

$$\begin{aligned} ||u_N(t,x)-u_M(t,x)||_{2,D}^2 &\leq \left\|\sum_{n=M}^{N}\bar{f}(n)\phi_n(x)h(t,\lambda_n)\right\|_{2,D}^2 \\ &\leq h(t,\lambda_{n_0})^2\sum_{n=n_0(\epsilon)}^{\infty}(\bar{f}(n))^2\leq\sum_{n=n_0(\epsilon)}^{\infty}(\bar{f}(n))^2<\epsilon. \end{aligned} \tag{4.15}$$

Thus, the series (4.12) converges in $L^2(D)$, uniformly in $t\geq 0$.

Step 2. Next we show that the initial function is defined as the L^2 limit of $u(t,x)$ as $t\to 0$, that is, we show that $t\to u(t,\cdot)\in C((0,\infty);L^2(D))$ and $u(t,\cdot)$ takes the

initial datum f in the sense of $L^2(D)$, i.e.,

$$||u(t,\cdot)-f||_{2,D}\to 0, \text{ as } t\to 0.$$

Since $h(t,\lambda)$ is the Laplace transform of E_t, it is completely monotone and non-increasing in $\lambda\geq 0$. Hence,

$$u(t,x)-f(x)=\sum_{n=1}^{\infty}\bar{f}(n)(h(t,\lambda_n)-1)\phi_n(x).$$

Fix $\epsilon\in(0,1)$ and choose $n_0=n_0(\epsilon)$ as in (4.14). Then,

$$\begin{aligned}
||u(t,\cdot)-f||_{2,D}^2 &= \sum_{n=1}^{\infty}(\bar{f}(n))^2(h(t,\lambda_n)-1)^2\\
&\leq \sum_{n=1}^{n_0(\epsilon)}(\bar{f}(n))^2(h(t,\lambda_n)-1)^2\\
&\quad+\sum_{n=n_0(\epsilon)+1}^{\infty}(\bar{f}(n))^2(h(t,\lambda_n)-1)^2\\
&\leq (1-h(t,\lambda_{n_0}))^2||f||_{2,D}+\epsilon
\end{aligned}$$

and now the claim follows, if $h(t,\lambda_n)\to 1$, as $t\to 0$.

This follows because $E_t\Rightarrow E_0$ in distribution as $t\to 0+$ and hence the Laplace transforms converge. To see that $E_t\Rightarrow E_0$, use the fact that $\{E_t\leq x\}=\{W_x\geq t\}$ which is (3.16) in [32]. Then for $x>0$ and $t_n\downarrow 0$,

$$P(E_{t_n}\leq x)=P(W_x\geq t_n)=1-P(W_x\leq t_n)\to 1-P(W_x\leq 0)=1,$$

since W_x has a density. As E_t also has a density on $[0,\infty)$ we see that $P(E_{t_n}\leq x)=0$, if $x<0$. Thus,

$$P(E_{t_n}\leq x)\to P(E_0\leq x)=I(x\geq 0)$$

at all continuity points of the limit. Note that $E_0=0$ almost surely, since $W_x>0$ almost surely for all $x>0$.

The continuity of $t\mapsto u(t,\cdot)$ in $L^2(D)$ at every point $t\in(0,\infty)$ can be proved in a similar fashion.

Step 3. A decay estimate for the solution $u(t,x)$ is obtained as follows. Using Parseval's identity, the fact that λ_n is increasing in n, and the fact that $h(t,\lambda_n)$ is non-increasing for $n\geq 1$, we get

$$||u(t,\cdot)||_{2,D}\leq h(t,\lambda_1)||f||_{2,D}.$$

Step 4. We next show that the series (4.12) defining $u(t,x)$ is the classical solution to (4.1), by proving its uniform and absolute convergence. We do this by showing that (4.13) is a Cauchy sequence in $L^\infty(D)$ uniformly in $t\geq 0$.

Applying the Green's second identity, we see that

$$\overline{\Delta f}(n) = \int_D \Delta f(x)\phi_n(x)dx = -\lambda_n \bar{f}(n).$$

Hence, $\Delta f = \sum_{n=1}^{\infty} -\lambda_n \phi_n(x)\bar{f}(n)$ is absolutely and uniformly convergent by assumption. Let $\epsilon > 0$. Since $\lambda_n \to \infty$ as $n \to \infty$, there exists an $n_0(\epsilon)$ so that for all $x \in D$,

$$\sum_{n=n_0(\epsilon)}^{\infty} |\bar{f}(n)||\phi_n(x)| \le \sum_{n=n_0(\epsilon)}^{\infty} \lambda_n |\bar{f}(n)||\phi_n(x)| < \epsilon. \tag{4.16}$$

This is possible since we have assumed that Δf has an expansion which is uniformly and absolutely convergent in $L^\infty(D)$. We will freely use the fact that the series defining f also converges absolutely and uniformly.
For $N > M > n_0(\epsilon)$ and $t \ge 0$ and $x \in D$,

$$\begin{aligned} |u_N(t,x) - u_M(t,x)| &= |\sum_{n=M}^{N} \phi_n(x)\bar{f}(n)h(t,\lambda_n)| \\ &\le \sum_{n=M}^{N} |\phi_n(x)||\bar{f}(n)| < \epsilon, \end{aligned} \tag{4.17}$$

since $h(t,\lambda) = \mathbb{E}(e^{-\lambda E_t}) \le 1$ for all $t \ge 0$ and $\lambda \ge 0$.

This shows that the sequence $u_N(t,x)$ is a Cauchy sequence in $L^\infty(D)$ and so has a limit in $L^\infty(D)$. Hence, the series in (4.12) is absolutely and uniformly convergent. Also, it follows that $u(t,x)$ satisfies the boundary conditions in (4.1).

Step 5. Next we show that the distributed-order fractional time derivative and the Laplacian Δ can be applied term by term in (4.12). As $h(t,\lambda)$ is bounded above by unity, we have

$$\begin{aligned} \left|\sum_{n=1}^{\infty} \bar{f}(n)h(t,\lambda_n)\Delta\phi_n(x)\right| &= \left|\sum_{n=1}^{\infty} \bar{f}(n)h(t,\lambda_n)\lambda_n\phi_n(x)\right| \\ &\le \sum_{n=1}^{\infty} |\phi_n(x)||\bar{f}(n)|\lambda_n < \infty, \end{aligned} \tag{4.18}$$

where the last inequality follows from the fact that the eigenfunction expansion of Δf converges absolutely and uniformly. Then the series

$$\sum_{n=1}^{\infty} \bar{f}(n)h(t,\lambda_n)\Delta\phi_n(x)$$

is absolutely convergent in $L^\infty(D)$ uniformly in $(0,\infty)$. Since $h(t,\lambda)$ is an eigenfunction of the distributed-order Caputo fractional derivative with $\mathbb{D}^{(\nu)}h(t,\lambda) = -\lambda h(t,\lambda)$

(see (3.10)), we have

$$\sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)\mathbb{D}^{(\nu)}h(t,\lambda_n) = \sum_{n=1}^{\infty} \bar{f}(n)h(t,\lambda_n)\Delta\phi_n(x).$$

As the two series are equal term-by-term and the series on the right converges absolutely and uniformly, the series on the left converges absolutely and uniformly too.

Now it is easy to check that the distributed-order fractional time derivative and Laplacian can be applied term by term in (4.12) to give

$$\begin{aligned}\mathbb{D}^{(\nu)}u(t,x) - \Delta u(t,x) & \\ &= \sum_{n=1}^{\infty} \bar{f}(n)\left[\phi_n(x)\mathbb{D}^{(\nu)}h(t,\lambda_n) - h(t,\lambda_n)\Delta\phi_n(x)\right] = 0,\end{aligned}$$

so that the PDE in (4.1) is satisfied. Thus, we conclude that u defined by (4.12) is a classical (strong) solution to (4.1).
Further, using Lemma 3.2, we get

$$\begin{aligned}\left|\frac{\partial u(t,x)}{\partial t}\right| &\leq \sum_{n=1}^{\infty} |\bar{f}(n)| \left|\frac{\partial h(t,\lambda_n)}{\partial t}\right| |\phi_n(x)| \\ &\leq k(t)\sum_{n=1}^{\infty} \lambda_n |\bar{f}(n)||\phi_n(x)| := k(t)g(x).\end{aligned}$$

Since Δf has absolutely and uniformly convergent series expansion with respect to $\{\phi_n : \ n \in \mathbb{N}\}$, we have $g \in L^\infty(D)$.

Thus, it follows, from the results obtained above, that $u \in \mathcal{H}_\Delta(D_\infty) \cap C_b(\bar{D}_\infty)$.

Step 6. We next show that $u \in C^1(\bar{D})$; this follows from the bounds in [20, Theorem 8.33] and the absolute and uniform convergence of the series defining f, namely,

$$|\phi_n|_{1,\alpha;D} \ \leq \ C(1+\lambda_n)\sup_D |\phi_n(x)|, \tag{4.19}$$

where $C = C(d, \lambda, \Lambda, \partial D)$ and λ is the constant in the definition of uniform ellipticity of L in (2.2) and Λ is the bound in (2.4). Here,

$$|u|_{k,\alpha;D} = \sup_{|\gamma|=k} [D^\gamma u]_{\alpha,D} + \sum_{j=0}^{k} \sup_{|\gamma|=j} \sup_{D} |D^\gamma u|, \quad k = 0, 1, 2, \cdots,$$

and

$$[D^\gamma u]_{\alpha,D} = \sup_{x,y\in D, x\neq y} \frac{|D^\gamma u(x) - D^\gamma u(y)|}{|x-y|^\alpha}$$

are norms on $C^{k,\alpha}(\bar{D})$. Hence,

$$\begin{aligned}
|u(t,.)|_{1,\alpha;D} &\leq C\sum_{n=1}^{\infty}|\bar{f}(n)|h(t,\lambda_n)(1+\lambda_n)\sup_D|\phi_n(x)| \\
&\leq C\sum_{n=1}^{\infty}|\bar{f}(n)|(1+\lambda_n)\sup_D|\phi_n(x)| \\
&\leq C\sum_{n=1}^{\infty}\sup_D|\phi_n(x)||\bar{f}(n)| + C\sum_{n=1}^{\infty}\lambda_n\sup_D|\phi_n(x)||\bar{f}(n)| < \infty.
\end{aligned}$$

Step 7. We obtain here the stochastic solution to (4.1), by inverting the ϕ_n-Laplace transform. After showing the absolute and uniform convergence of the series defining u, we can use a Fubini-Tonelli type argument to interchange order of summation and integration in the following, together with (4.12) and (4.9), to get a stochastic representation of the solution as

$$\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty}\phi_n(x)\int_0^{\infty}\overline{[T_D(l)f]}(n)g(t,l)dl \\
&= \int_0^{\infty}\left[\sum_{n=1}^{\infty}\phi_n(x)\bar{f}(n)e^{-l\lambda_n}\right]g(t,l)dl \\
&= \int_0^{\infty}T_D(l)f(x)g(t,l)dl \\
&= E_x[f(X(E_t))I(\tau_D(X) > E_t)]. \qquad (4.20)
\end{aligned}$$

The last equality follows from a simple conditioning argument and using (2.7).

Step 8. Finally, we prove the uniqueness. Let $u_i, i = 1, 2$, be two solutions of (4.1) with initial data $u_i(0,x) = f(x)$ and Dirichlet boundary condition $u_i(t,x) = 0$ for $x \in \partial D$. Then $U = u_1 - u_2$ is also a solution of (4.1) with zero initial data and zero boundary value. Taking ϕ_n-transform on both sides of (4.1) we get

$$\mathbb{D}^{(\nu)}\bar{U}(t,n) = -\lambda_n\bar{U}(t,n), \quad \bar{U}(0,n) = 0,$$

and then $\bar{U}(t,n) = 0$ for all $t > 0$ and all $n \geq 1$. This implies that $U(t,x) = 0$ in the sense of L^2 functions, since $\{\phi_n : n \geq 1\}$ forms a complete orthonormal basis for $L^2(D)$. Hence, $U(t,x) = 0$ for all $t > 0$ and almost all $x \in D$. Since U is a continuous function on D, we have $U(t,x) = 0$ for all $(t,x) \in [0,\infty) \times D$, thereby proving the uniqueness. $\square$

Corollary 4.2. *The solution in Theorem (4.1) also has the following representation:*

$$u(t,x) = E_x[f(X(E_t))I(\tau_D(X) > E_t)] = E_x[f(X(E_t))I(\tau_D(X(E)) > t)].$$

Proof. The argument is similar to [34, Corollary 3.2] and so we only sketch the proof. Given a continuous stochastic process X_t on $\mathbb{R}^d$, and an interval $I \subset [0, \infty)$, we denote $X(I) = \{X_u : u \in I\}$. Since the domain D is open and X_u is continuous, it follows that

$$\{\tau_D(X) > t\} = \{X([0,t]) \subset D\}.$$

Next note that, since E_t is continuous and monotone nondecreasing, $E([0,t]) = [0, E_t]$. Finally, we observe that

$$\begin{aligned}\{\tau_D(X(E)) > t\} &= \{X(E([0,t]) \subset D\} \\ &= \{X([0,E_t]) \subset D\} = \{\tau_D(X) > E_t\}\end{aligned}$$

which completes the proof. □

Remark 4.3. If we time-change Brownian motion $B(t)$ using a nondecreasing stable Lévy process W_t, then the conclusions of Corollary 4.2 do not hold, since the stable subordinator does not have continuous sample paths; see, for example, Song and Vondraček [46]. In our case, killing the process $B(t)$ and then applying the time change E_t is the same as applying the time change and then killing, since the sample paths of E_t are continuous.

The next result establishes existence of strong solutions of distributed-order fractional Cauchy problems (3.7) with $L = \Delta$.

Corollary 4.4. *Let $f \in C_c^{2k}(D)$ be a $2k$-times continuously differentiable function of compact support in D. If $k > 1 + 3d/4$, then (4.1) has a classical (strong) solution. In particular, if $f \in C_c^\infty(D)$, then the solution of (4.1) is in $C^\infty(D)$.*

Proof. By Example 2.1.8 of [16], $|\phi_n(x)| \le (\lambda_n)^{d/4}$. Also, from Corollary 6.2.2 of [17], we have $\lambda_n \sim n^{2/d}$.
Applying Green's second identity k-times, we get

$$\overline{\Delta^k f}(n) = \int_D \Delta^k f(x)\phi_n(x)dx = (-\lambda_n)^k \bar{f}(n). \tag{4.21}$$

Using Cauchy-Schwartz inequality and the fact $f \in C_c^{2k}(D)$, we get

$$\overline{\Delta^k f}(n) \le \left[\int_D (\Delta^k f(x))^2 dx\right]^{1/2} \left[\int_D (\phi_n(x))^2 dx\right]^{1/2} = \left[\int_D (f^{2k}(x))^2 dx\right]^{1/2} = c_k,$$

where c_k is a constant independent of n.
This and (4.21) give $|\bar{f}(n)| \le c_k(\lambda_n)^{-k}$.
Since

$$\Delta f(x) = \sum_{n=1}^{\infty} -\lambda_n \bar{f}(n)\phi_n(x),$$

to get the absolute and uniform convergence of the series defining Δf, we consider

$$\sum_{n=1}^{\infty}\lambda_n|\phi_n(x)||\bar{f}(n)| \le \sum_{n=1}^{\infty}(\lambda_n)^{d/4+1}c_k(\lambda_n)^{-k}$$
$$\le c_k\sum_{n=1}^{\infty}(n^{2/d})^{d/4+1-k} = c_k\sum_{n=1}^{\infty}n^{1/2+2/d-2k/d}$$

which is finite if $(\frac{2k}{d}-\frac{2}{d}-\frac{1}{2})>1$, i.e., if $k>1+\frac{3}{4}d$. $\square$

Remark 4.5. In an interval $(0,M)\subset\mathbb{R}$, eigenfunctions and eigenvalues are explicitly known. Eigenvalues of the Laplacian on $(0,M)$ are $(n\pi/M)^2$, and the corresponding eigenfunctions are $\sin(n\pi x/M)$, for $n=1,2,\cdots$. The form of the solution in (4.12) on a bounded interval $(0,M)$ in $\mathbb{R}$ was obtained by [1, 37]. Agrawal [1] worked with single-order fractional Cauchy problem with Dirichlet boundary conditions. Naber [37] considered Dirichlet and Neumann boundary conditions in one space dimension.

Recall that $D_\infty=(0,\infty)\times D$ and define now

$$\mathcal{H}_L(D_\infty)=\{u:D_\infty\to\mathbb{R}:\quad Lu(t,x)\in C(D_\infty)\};$$
$$\mathcal{H}_L^b(D_\infty)=\mathcal{H}_L(D_\infty)\cap\{u:|\partial_t u(t,x)|\le k(t)g(x),\quad g\in L^\infty(D),\ t>0\}$$

where $k(t)$ is defined in (3.17). The following result extends Theorem 4.1 to general uniformly elliptic second-order operators.

Theorem 4.6. *Let D be a bounded domain with $\partial D\in C^{1,\alpha}$ for some $0<\alpha<1$, and suppose that L is given by (2.1) with $a_{ij}\in C^\alpha(\bar{D})$. Let $\{X(t)\}$ be a continuous Markov process with generator L, and $T_D(t)$ the killed semigroup corresponding to the process $\{X(t)\}$ in D. Let E_t be the inverse (3.3) of the subordinator W_t, independent of $\{X(t)\}$, with Lévy measure (3.5). Suppose that $\mu(d\beta)=p(\beta)d\beta$, as in Lemma 3.2, and $\mathbb{D}^{(\nu)}$ is the distributed-order fractional derivative defined by (3.8). Then, for any $f\in D(L_D)\cap C^1(\bar{D})\cap C^2(D)$ for which the eigenfunction expansion (of Lf) with respect to the complete orthonormal basis $\{\psi_n:\ n\ge 1\}$ converges uniformly and absolutely, the (classical) solution of*

$$\begin{aligned}\mathbb{D}^{(\nu)}u(t,x) &= Lu(t,x),\quad x\in D,\ t\ge 0;\\ u(t,x) &= 0,\ x\in\partial D,\ t\ge 0;\\ u(0,x) &= f(x),\ x\in D,\end{aligned}\tag{4.22}$$

for $u\in\mathcal{H}_L^b(D_\infty)\cap C_b(\bar{D}_\infty)\cap C^1(\bar{D})$, is given by

$$\begin{aligned}u(t,x) &= E_x[f(X(E_t))I(\tau_D(X)>E_t)]=E_x[f(X(E_t))I(\tau_D(X(E))>t)]\\ &= \int_0^\infty T_D(l)f(x)g(t,l)dl=\sum_0^\infty \bar{f}(n)\psi_n(x)h(t,\mu_n),\end{aligned}\tag{4.23}$$

where $h(t,\mu)=\mathbb{E}(e^{-\mu E_t})=\int_0^\infty e^{-x\mu}g(t,x)\,dx$ is the Laplace transform of E_t.

Proof. Let $u(t,x) = G(t)F(x)$ be a solution of (4.22). Substituting in the PDE (4.22) leads to

$$F(x)\mathbb{D}^{(\nu)}G(t) = G(t)LF(x)$$

and now dividing both sides by $G(t)F(x)$, we obtain

$$\frac{\mathbb{D}^{(\nu)}G(t)}{G(t)} = \frac{LF(x)}{F(x)} = -\mu.$$

That is,

$$\mathbb{D}^{(\nu)}G(t) = -\mu G(t),\ t > 0; \tag{4.24}$$

$$LF(x) = -\mu F(x),\ x \in D,\ F|_{\partial D} = 0. \tag{4.25}$$

Problem (4.25) is solved by an infinite sequence of pairs (μ_n, ψ_n), $n \geq 1$, where $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$ is a sequence of numbers such that $\mu_n \to \infty$, as $n \to \infty$, and ψ_n is a sequence of functions that form a complete orthonormal set in $L^2(D)$ (cf. (2.5)). In particular, the initial function f regarded as an element of $L^2(D)$ can be represented as

$$f(x) = \sum_{n=1}^{\infty} \bar{f}(n)\psi_n(x). \tag{4.26}$$

An application of the Parseval identity yields

$$||f||_{2,D} = \sum_{n=1}^{\infty} (\bar{f}(n))^2. \tag{4.27}$$

Using the μ_n determined by (4.25) and recalling from Lemma 3.1 that $h(t, \mu_n)$ solves (4.24) with $\mu = \mu_n$, we obtain

$$G(t) = G_0(n)h(t, \mu_n),$$

where $G_0(n)$ is selected to satisfy the initial condition f. We will show that

$$u(t,x) = \sum_{n=1}^{\infty} \bar{f}(n)h(t, \mu_n)\psi_n(x) \tag{4.28}$$

solves the PDE (4.22).

Define approximate solutions of the form

$$u_N(t,x) = \sum_{n=1}^{N} G_0(n)h(t, \mu_n)\psi_n(x),\ G_0(n) = \bar{f}(n). \tag{4.29}$$

Step 1. Following the proof of Theorem 4.1, it can be shown that the sequence $\{u_N(t,\cdot)\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2(D) \cap L^\infty(D)$, uniformly in $t \in [0, \infty)$.

Hence, the solution to (4.22) is given formally by

$$u(t,x) = \sum_{n=1}^{\infty} \bar{f}(n) h(t,\mu_n) \psi_n(x). \tag{4.30}$$

Step 2. It follows again, as in the proof of Theorem 4.1, the series defining u and Lu converge absolutely and uniformly so that we can apply the fractional time derivative and uniformly elliptic operator L term by term to show that u defined by (4.30) is indeed a classical solution to (4.22).

Step 3. The stochastic representation of the solution, as given in (4.23), also follows in a similar manner and hence we omit the details.

Step 4. We have also a decay estimate for u as in Theorem 4.1, namely,

$$||u(\cdot,t)||_{2,D} \le h(t,\mu_1)||f||_{2,D}.$$

The uniqueness of the solution can be proved as before. $\square$

5. Extensions and open questions

In this section, we derive some general conditions on the mixing distribution $\mu(d\beta)$ in (3.8) that are sufficient to obtain classical solutions to the distributed-order fractional Cauchy problem (3.7) on bounded domains, as in Theorems 4.1 and 4.6. In particular, we remove the assumption that $\mu(d\beta) = p(\beta)d\beta$ to allow atoms. Then we discuss related literature, and some open questions.

Lemma 5.1. *Suppose μ is a finite measure with supp$(\mu) \subset (0,1)$ that satisfies* (3.9). *Assume also that $|\partial_t h(t,\lambda)| \le b(\lambda)k_e(t)$ for some functions b and k_e satisfying the condition*

$$b(\lambda)\int_0^1 \int_0^t \frac{k_e(s)ds}{(t-s)^{\beta}} d\mu(\beta) < \infty, \tag{5.1}$$

for $t,\lambda > 0$. Then $h(t,\lambda) = \mathbb{E}(e^{-\lambda E_t})$ is a classical solution of the eigenvalue problem

$$\mathbb{D}^{(\nu)} h(t,\lambda) = -\lambda h(t,\lambda); \quad h(0,\lambda) = 1. \tag{5.2}$$

Proof. The proof follows from Lemma 3.1, and the fact that (5.1) is a sufficient condition for $\mathbb{D}^{(\nu)}h(t,\lambda)$ to be defined as a classical function. $\square$

Suppose that $|\partial_t h(t,\lambda)| \le b(\lambda)k_e(t)$, where b and k_e satisfy, in addition to (5.1),

$$k_e(t)\sum_{n=1}^{\infty} b(\lambda_n)\bar{f}(n)|\phi_n(x)| < \infty. \tag{5.3}$$

It is assumed here that the above series converges absolutely and uniformly for $t > 0$.

Let $\beta \in (0,1)$, $D_\infty = (0,\infty) \times D$ and define

$$\mathcal{H}_L(D_\infty) = \{u : D_\infty \to \mathbb{R} : \quad Lu(t,x) \in C(D_\infty)\};$$
$$\mathcal{H}_L^{b,e}(D_\infty) = \mathcal{H}_L(D_\infty) \cap \{u : |\partial_t u(t,x)| \le k_e(t) g(x), \quad g \in L^\infty(D), \ t > 0\},$$

where k_e and b satisfy (5.1) and (5.3). The following result extends Theorem 4.6 to allow atoms in the mixing measure $\mu(d\beta)$.

Theorem 5.2. *Let D be a bounded domain with $\partial D \in C^{1,\alpha}$ for some $0 < \alpha < 1$, and suppose that L is given by (2.1) with $a_{ij} \in C^\alpha(\bar{D})$. Let $\{X(t)\}$ be a continuous Markov process with generator L, and $T_D(t)$ be the killed semigroup corresponding to the process $\{X(t)\}$ in D. Let E_t be the inverse (3.3) of the subordinator W_t, independent of $\{X(t)\}$, with Lévy measure (3.5). Let $f \in D(L_D) \cap C^1(\bar{D}) \cap C^2(D)$ for which the eigenfunction expansion (of Lf) with respect to the complete orthonormal basis $\{\psi_n : \ n \ge 1\}$ converges uniformly and absolutely. Then the (classical) solution of*

$$\begin{aligned} \mathbb{D}^{(\nu)} u(t,x) &= Lu(t,x), \quad x \in D, \ t \ge 0; \\ u(t,x) &= 0, \ x \in \partial D, \ t \ge 0; \\ u(0,x) &= f(x), \ x \in D, \end{aligned} \tag{5.4}$$

for $u \in \mathcal{H}_L^{b,e}(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D})$, with the distributed order fractional derivative $\mathbb{D}^{(\nu)}$ defined by (3.8), is given by

$$\begin{aligned} u(t,x) &= E_x[f(X(E_t)) I(\tau_D(X) > E_t)] = E_x[f(X(E_t)) I(\tau_D(X(E)) > t)] \\ &= \int_0^\infty T_D(l) f(x) g(t,l) dl = \sum_0^\infty \bar{f}(n) \psi_n(x) h(t,\mu_n), \end{aligned} \tag{5.5}$$

where $h(t,\mu) = \mathbb{E}(e^{-\mu E_t}) = \int_0^\infty e^{-x\mu} g(t,x)\, dx$ is the Laplace transform of E_t.

Proof. The proof follows the same steps as in the proof of Theorems 4.1 and 4.6 and using the properties (5.1) and (5.3) of the function $k_e(t)$. □

Remark 5.3. Let $\mu(d\beta) = \sum_{j=1}^N c_j^{\beta_j} (\Gamma(1-\beta_j))^{-1} \delta_{\beta_j}(\beta) d\beta$, for $0 < \beta_1 < \beta_2 < \cdots < \beta_N < 1$. In this case, the subordinator is $W_t = \sum_{j=1}^N c_j W_t^{\beta_j}$ for independent stable subordinators $W_t^{\beta_j}$, for $j = 1, \cdots, N$. In this case, the functions $k_e(t)$ and $b(\lambda)$ that satisfy (5.3), (5.1) and

$$|\partial_t h(t,\lambda)| \le b(\lambda) k_e(t)$$

are $k_e(t) = (c_j^{\beta_j} \sin(\beta_j \pi))^{-1} (t^{\beta_j - 1})$ for all $j = 1, \cdots, N$ and $b(\lambda) = \lambda$, respectively. The proof of this fact follows the same steps as in the proof of equation (2.19) in [27] using the properties of $\mu(\beta)$. Hence, in this case Theorem 5.2 applies and we have a classical solution of (5.4) given by (5.5).

To conclude this paper, we discuss the related literature and some open problems. Given a uniformly bounded, strongly continuous semigroup $T(t)$ with generator L on some Banach space H, the Cauchy problem $\partial u(t,x)/\partial t = Lu(t,x)$ with $u(0,x) = f(x)$ has solution $u(t,x) = T(t)f(x)$ for any initial condition $f \in H$ [3]. One important special case is the pseudo-differential operator

$$
\begin{aligned}
Lu(t,x) = &\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u(t,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u(t,x)}{\partial x_i} \\
&+ \int_{y\neq 0}\left(u(t,x-y) - u(t,x) + \frac{\sum_{i=1}^{d}\frac{\partial u(t,x)}{\partial x_i}y_i}{1+\sum_{i=1}^{d}y_i^2}\right)\phi(x,dy)
\end{aligned}
\tag{5.6}
$$

that appears in the backward equation of a Markov process $X(t)$ [24, 44]. The probability distribution of the Markov process $X(t)$ solves the forward equation, which is the Cauchy problem with the adjoint of the generator L. The integral term in (5.6) represents a jump diffusion (e.g., a stable process). For stable generators, the explicit connection with stochastic differential equations driven by a stable Lévy process was established by Zhang et al. [48] and Chakraborty [11]. In that case, the integral term in (5.6) can be written in terms of fractional derivatives in the space variable.

The solution (4.2) to the fractional Cauchy problem $\partial^\beta u(t,x)/\partial t^\beta = Lu(t,x)$ for $0 < \beta < 1$ was established by Baeumer and Meerschaert [4] in the general Banach space setting. Baeumer et al. [5] and Nane [38] specialized to Markov processes with generator (5.6), and established a connection to iterated Brownian motion [2, 8, 9, 18]. Hahn et al. [22] developed the connection with stochastic differential equations driven by a time-changed Lévy process $X(E_t)$ for generators (5.6), so that their result includes jump diffusions on $\mathbb{R}^d$. Their results extend those of [11, 48] to distributed-order fractional Cauchy problems on $\mathbb{R}^d$. Hahn et al. [22] also give the integral solution (4.2) as in [4, 5]. Kochubei [27] provides strong solutions of distributed order fractional Cauchy problems on $\mathbb{R}^d$ in the case $L = \Delta$. Meerschaert and Scheffler [33] discuss generalized Cauchy problems of the form $\psi_W(\partial/\partial t)u(t,x) = Lu(t,x) + \delta(x)\psi_W(t,\infty)$, where $\psi_W(s)$ is the Laplace exponent of a nondecreasing Lévy process (subordinator) whose Lévy measure ϕ_W has infinite total mass, and L is the generator of another Lévy process. This reduces to the distributed order fractional Cauchy problem (3.7) in the special case when (3.4) holds. As in Section 3, the paper [33] shows that the density $u(t,x)$ of the CTRW scaling limit $A(E_t)$ solves the generalized Cauchy problem, when E_t is the inverse of the subordinator W_t with $\mathbb{E}[e^{-sW_t}] = e^{-t\psi_W(s)}$. Strong solutions of generalized Cauchy problems on $\mathbb{R}^d$, including fractional or distributed-order Cauchy problems, seem to be an open problem.

For bounded domains, the general results of [4] remain valid, so that the solution formula (4.2) still holds in the appropriate Banach space. The results of this paper provide strong solutions in that case, so long as L generates a diffusion without jumps. To the best of our knowledge, the construction of strong solutions for jump

diffusions remains a challenging open problem. Eigenvalue expansions can be found explicitly in some special cases. The main technical difficulty is to obtain regularity of the eigenfunctions, or at least sharp bounds, for the generator (5.6) in the case of jump diffusions on bounded domains. See Chen et al. [14] for a recent study on this problem. One explicit example is to take $L = -(-\Delta)^{\alpha/2}$ for $0 < \alpha < 2$, the classical fractional power of the Laplacian [23], which generates a spherically symmetric stable Lévy process. This results from (5.6) with $a = b = 0$ and $\phi(x, dy) = C_{d,\alpha}\|y\|^{-\alpha-1}dy$, where $C_{d,\alpha}$ is a constant that depends on the stable index α and the dimension d of the space, see for example [29]. This is a type of fractional derivative in space, called the Riesz fractional derivative of order α. Some results for this case are available in Chen and Song [12], Chen et al. [13] and Song and Vondraček [46].

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